LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION - **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2013**

MT 1816 - REAL ANALYSIS

Date : 08/11/2013 Dept. No. Max. : 100 Marks
Time : 1:00 - 4:00

Answer **ALL** Questions. All questions carry equal marks.

1. a) (i) Prove that if P^* is a refinement of P, then $L(P,f,\alpha) \le L(P^*,f,\alpha)$ and $U(P,f,\alpha) \ge U(P^*,f,\alpha)$. (5)

OR

(ii) If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on [a, b], then prove that $fg \in \Re(\alpha)$. (5)

b) (i) Assume α increases monotonically and α' □ $\Re$ on [a, b]. Let f be a bounded real function on [a, b]. Then prove that $f \in \Re(\alpha)$ if and only if $f\alpha'$ □ □ and in that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

(ii) State and prove the fundamental theorem of calculus. (9+6)

OR

(iii) Suppose f is bounded on [a, b], f has only finitely many points of discontinuity on [a, b] and α is continuous at every point at which f is discontinuous. Then prove that $f \in$ □(α).

(iv) Let $f \in$ □(α) on [a, b], $m \le f \le M$, φ be continuous on [m, M] and $h(x) = \varphi(f(x))$ on [a,b]. Then prove that $h \in$ □(α) on [a, b]. (8+7)

2. (a)(i) Illustrate with an example that the limit of the integral need not be equal to the integral of the limit. (5)

OR

(ii) Prove that the sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \ge N$, $n \ge N$, $x \varepsilon E$ implies $|f_n(x) - f(x)| \le \varepsilon$. (5)

(P.T.O)

(b) (i) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n$ $n = 1,2,3,...$, then prove that $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

(ii) Let α be monotonically increasing on $[a,b]$, $f_n \varepsilon$ □(α) on $[a,b]$, for $n = 1,2,3,...$ and suppose $f_n \to f$ on $[a,b]$. Then prove that $f \varepsilon$ □(α) on $[a,b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$. (8+7)

OR

(iii) State and prove Stone- Weierstrass theorem. (15)

3. a) (i) Let $S = \{\varphi_0, \varphi_1, \varphi_2, ...\}$, where $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}$, $\varphi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}$ and $\varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}$, for n = 1, 2…. Prove that S is orthnormal on any interval of length 2 π. (5)

OR

(ii) Let $= \{\varphi_0, \varphi_1, \varphi_2, ...\}$ be orthnormal on I and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows: $s_n(x) = \sum_{k=0}^{\infty} c_k\varphi_k(x)$, $t_n(x) = \sum_{k=0}^{\infty} b_k\varphi_k(x)$

where $c_k = (f, \varphi_k(x))$ for k = 0, 1, 2… and b_0, b_1, b_2 … are arbitrary complex numbers. Then for each n, prove that $\|f - s_n\| \le \|f - t_n\|$.
(5)

b) (i) State and prove Riesz-Fischer theorem.
(ii) State and prove Riemann-Lebesgue lemma. (8+7)

OR

(iii) If g is of bounded variation on $[0, \delta]$, then prove that $\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^{\delta} g(t)\frac{\sin \alpha t}{t}\,dt = g(0+)$.

(iv) Assume that $f \in L[0,2\pi]$ and suppose that f is periodic with period 2π. Let $\{s_n\}$ denote the sequence of partial sums of the Fourier series generated by f, $s_n(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty}(a_k \cos kx + (b k \sin kx)$, n=1,2,…. Then prove that
$s_n(x) = \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t)\,dt$ where D_n is called Dirichlet's Kernel. (8+7)

(P.T.O)

4.(a) (i) Let *r* be a positive integer. If a Vector space *X* is spanned by a set of *r* vectors, then prove that $\dim X \le r$.
(5) OR

(ii) Prove that a linear operator A on a finite dimensional vector space X is one-to-one if and only if the range of A is all of X. (5)

(b) (i) Suppose E is an open set in R^n, f maps E into R^m, f is differentiable at $x_0 \varepsilon E$, g maps an open set containing $f(E)$ into R^k, and g is differentiable at $f(x_0)$. Then prove that the mapping F of E into R^k defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.
(15) OR

(ii) State and prove Inverse function theorem. (15)

5. (a)(i) Graph the ellipse given by $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Using the graphical approach, determine parts of the graph that have inverses and algebraic approach, find invertible formulas and cases converting x to y. (5)

OR

(ii) Define heat flow and the heat equation. (5)

(b) (i) Derive D' Alembert's approach toward characterizing solutions of the one dimensional wave equation.
(ii) Derive the solution to the heat equation. (6+9)

OR

(iii) Use matrix notation to solve

$$\frac{\partial f}{\partial x}\bullet 1 + \frac{\partial f}{\partial y}\bullet 0 + \frac{\partial f}{\partial u}\bullet\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\bullet\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial f}{\partial y}\bullet 0 + \frac{\partial f}{\partial y}\bullet 1 + \frac{\partial f}{\partial u}\bullet\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\bullet\frac{\partial v}{\partial y} = 0$$
$$\frac{\partial g}{\partial x}\bullet 1 + \frac{\partial g}{\partial y}\bullet 0 + \frac{\partial g}{\partial u}\bullet\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\bullet\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial g}{\partial y}\bullet 0 + \frac{\partial g}{\partial y}\bullet 1 + \frac{\partial g}{\partial u}\bullet\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\bullet\frac{\partial v}{\partial y} = 0$$

What is the criterion needed for the existence of the inverse matrix? (9)

(iv) Convert the river coordinates (u, v) into geographic (x, y) coordinates. (6)